On Characterization Canal Surfaces around Timelike Horizontal Biharmonic Curves in Lorentzian Heisenberg Group Heis^3

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Z. Naturforsch. **66a,** 441 – 449 (2011); received June 30, 2010 / revised December 9, 2010

In this paper, we describe a new method for constructing a canal surface surrounding a timelike horizontal biharmonic curve in the Lorentzian Heisenberg group Heis^3. Firstly, we characterize timelike biharmonic curves in terms of their curvature and torsion. Also, by using timelike horizontal biharmonic curves, we give explicit parametrizations of canal surfaces in the Lorentzian Heisenberg group Heis^3.

Key words: Canal Surface; Biharmonic Curve; Heisenberg Group.
Mathematics Subject Classification 2000: 31B30, 58E20

1. Introduction

Canal surfaces are very useful for representing long thin objects, for instance, poles, 3D fonts, brass instruments or internal organs of the body in solid modelling. It includes natural quadrics (cylinder, cone, and sphere), revolute quadrics, tori, pipes, and Dupin cyclide. Also, canal surfaces are among the surfaces which are easier to describe both analytically and operationally. They are still under active investigation, both for finding best parameterizations (see, for instance, [1 – 6]) or for application in different fields (for instance in medicine, see [7]).

We remind that, if C is a space curve, a tubular surface associated to this curve is a surface swept by a family of spheres of constant radius (which will be the radius of the tube), having the center on the given curve. Alternatively, as we shall see in the next section, for them we can construct quite easily a parameterization using the Frenet frame associate to the curve. The tubular surfaces are used quite often in computer graphics, but we think they deserve more attention for several reasons. For instance, there is the problem of representing the curves themselves. Usually, the space curves are represented by using solids rather then tubes. There are, today, several very good computer algebra system (such as Maple, or Mathematica) which allow the vizualisation of curves and surfaces in different kind of representations.

The aim of this paper is to study a canal surface surrounding a timelike horizontal biharmonic curve in the Lorentzian Heisenberg group Heis^3.

Let (M,g) and (N,h) be Lorentzian manifolds and $\phi : M \to N$ a smooth map. Denote by ∇^{ϕ} the connection of the vector bundle ϕ^*TN induced from the Levi–Civita connection ∇^h of (N,h). The *second fundamental form* $\nabla\,\mathrm{d}\phi$ is defined by

$$(\nabla\,\mathrm{d}\phi)(X,Y) = \nabla^{\phi}_X \mathrm{d}\phi(Y) - \mathrm{d}\phi(\nabla_X Y),\\ X,Y \in \Gamma(TM).$$

Here ∇ is the Levi–Civita connection of (M,g). The tension field $\tau(\phi)$ is a section of ϕ^*TN defined by

$$\tau(\phi) = \mathrm{tr}\nabla\,\mathrm{d}\phi. \qquad (1)$$

A smooth map ϕ is said to be *harmonic* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*:

$$E(\phi) = \frac{1}{2}\int h(\mathrm{d}\phi, \mathrm{d}\phi)\,\mathrm{d}v_g$$

over every compact region of M. Now let $\phi\colon M \to N$ be a harmonic map. Then the Hessian $\mathcal{H}$ of E is given by

$$\mathcal{H}_{\phi}(V,W) = \int h(\mathcal{J}_{\phi}(V),W)\,\mathrm{d}v_g,\\ V,W \in \Gamma(\phi^*TN).$$

Here the *Jacobi operator* $\mathcal{J}_\phi$ is defined by

$$\mathcal{J}_\phi(V) := \Delta^\phi V - \mathcal{R}_\phi(V), \quad V \in \Gamma(\phi^* TN), \tag{2}$$

$$\Delta^\phi := -\sum_{i=1}^{m}\left(\nabla_{e_i}^{\phi}\nabla_{e_i}^{\phi} - \nabla_{\nabla_{e_i}e_i}^{\phi}\right), \qquad \mathcal{R}_\phi(V) = \sum_{i=1}^{m} R^N(V, \mathrm{d}\phi(e_i))\,\mathrm{d}\phi(e_i), \tag{3}$$

where R^N and $\{e_i\}$ are the Riemannian curvature of N and a local orthonormal frame field of M, respectively, [8 – 15].

Let $\phi : (M,g) \to (N,h)$ be a smooth map between two Lorentzian manifolds. The *bienergy* $E_2(\phi)$ of ϕ over compact domain $\Omega \subset M$ is defined by

$$E_2(\phi) = \int_\Omega h(\tau(\phi), \tau(\phi))\,\mathrm{d}v_g.$$

A smooth map $\phi : (M,g) \to (N,h)$ is said to be *biharmonic* if it is a critical point of the $E_2(\phi)$.

The section $\tau_2(\phi)$ is called the *bitension field* of ϕ and the Euler–Lagrange equation of E_2 is

$$\tau_2(\phi) := -\mathcal{J}_\phi(\tau(\phi)) = 0. \tag{4}$$

Biharmonic functions are utilized in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them. In linear elasticity, if the equations are formulated in terms of displacements for two-dimensional problems then the introduction of a stress function leads to a fourth-order equation of biharmonic type. For instance, the stress function is proved to be biharmonic for an elastically isotropic crystal undergoing phase transition, which follows spontaneous dilatation. Biharmonic functions also arise when dealing with transverse displacements of plates and shells. They can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions arise in fluid dynamics, particularly in Stokes flow problems (i.e., low-Reynolds-number flows). There are many applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [2, 16]). Fluid flow through a narrow pipe or channel, such as that used in micro-fluidics, involves low Reynolds number. Seepage flow through cracks and pulmonary alveolar blood flow can also be approximated by Stokes flow. Stokes flow also arises in flow through porous media, which have been long applied by civil engineers to groundwater movement. The industrial applications include the fabrication of microelectronic components, the effect of surface roughness on lubrication, the design of polymer dies and the development of peristaltic pumps for sensitive viscous materials. In natural systems, creeping flows are important in biomedical applications and studies of animal locomotion.

In [17] the authors completely classified the biharmonic submanifolds of the three-dimensional sphere, while in [18] there were given new methods to construct biharmonic submanifolds of codimension greater than one in the n-dimensional sphere. The biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. The proper biharmonic curves on Riemannian surfaces were studied in [19]. Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in three-dimensional Sasakian space forms [20]. Then, Sasahara gave in [21] the explicit representation of the proper biharmonic Legendre surfaces in five-dimensional Sasakian space forms.

In this paper, we describe a new method for constructing a canal surface surrounding a timelike horizontal biharmonic curve in the Lorentzian Heisenberg group Heis^3. Firstly, we characterize timelike biharmonic curves in terms of their curvature and torsion. Also, by using timelike horizontal biharmonic curves, we give explicit parametrizations of canal surfaces in the Lorentzian Heisenberg group Heis^3.

2. The Lorentzian Heisenberg Group Heis^3

The Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, partial differential equations (PDEs) or even quantum mechanics, where it was initially defined as a group of 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

with the usual multiplication rule.

We will use the following complex definition of the Heisenberg group.

$$\mathrm{Heis}^3 = \mathbb{C} \times \mathbb{R} = \left\{ (w,z) : w \in \mathbb{C}, z \in \mathbb{R} \right\}$$

with

$$(w,z) * (\tilde{w},\tilde{z}) = \left(w + \tilde{w}, z + \tilde{z} + \mathrm{Im}(\langle w, \tilde{w} \rangle)\right),$$

where $\langle , \rangle$ is the usual Hermitian product in $\mathbb{C}$.

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$.

Let $a = (w_1,z_1)$, $b = (w_2,z_2)$, and $c = (w_3,z_3)$. The commutator of the elements $a,b \in \mathrm{Heis}^3$ is equal to

$$\begin{aligned}
[a,b] &= a * b * a^{-1} * b^{-1} \\
&= (w_1,z_1) * (w_2,z_2) * (-w_1,-z_1) * (-w_2,-z_2) \\
&= (w_1 + w_2 - w_1 - w_2, z_1 + z_2 - z_1 - z_2) \\
&= (0,\alpha),
\end{aligned}$$

where $\alpha \neq 0$ in general. For example

$$[(1,0),(i,0)] = (0,2) \neq (0,0).$$

Which shows that Heis^3 is not Abelian.

On the other hand, for any $a,b,c \in \mathrm{Heis}^3$, their double commutator is

$$\begin{aligned}
[[a,b],c] &= \left[(0,\alpha),(w_3,z_3)\right] \\
&= (0,0).
\end{aligned}$$

This implies that Heis^3 is a nilpotent Lie group with nilpotency 2.

The left-invariant Lorentz metric on Heis^3 is

$$g = -\mathrm{d}x^2 + \mathrm{d}y^2 + (x\mathrm{d}y + \mathrm{d}z)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z},\ \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \quad (5)$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2,\mathbf{e}_3] = 2\mathbf{e}_1,\ [\mathbf{e}_3,\mathbf{e}_1] = 0,\ [\mathbf{e}_2,\mathbf{e}_1] = 0,$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = 1,\ g(\mathbf{e}_3,\mathbf{e}_3) = -1. \quad (6)$$

Proposition 2.1. *For the covariant derivatives of the Levi–Civita connection of the left-invariant metric g defined above, the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \quad (7)$$

where the (i,j)*-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

We adopt the following notation and sign convention for the Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)W,Z).$$

Moreover, we put

$$R_{ijk} = R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

where the indices i, j, k, and l take the values 1, 2, and 3.

$$R_{121} = \mathbf{e}_2,\ R_{131} = \mathbf{e}_3,\ \ R_{232} = -3\mathbf{e}_3$$

and

$$R_{1212} = -1,\ R_{1313} = 1,\ R_{2323} = -3. \quad (8)$$

3. Timelike Biharmonic Curves in the Lorentzian Heisenberg Group Heis³

The biharmonic equation for the curve γ reduces to

$$\tau_2(\gamma) = \nabla^3_{\mathbf{T}(s)}\mathbf{T}(s) - R\left(\mathbf{T}(s), \nabla_{\mathbf{T}(s)}\mathbf{T}(s)\right)\mathbf{T}(s) = 0,$$

that is, γ is called a biharmonic curve if it is a solution of the above equation.

An arbitrary curve $\gamma : I \to \mathrm{Heis}^3$ is spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Let $\gamma : I \to \mathrm{Heis}^3$ be a unit speed timelike curve and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ are Frenet vector fields, then Frenet formulas are as follows:

$$\begin{aligned}
&\nabla_{\mathbf{T}(s)}\mathbf{T}(s) = \kappa_1(s)\mathbf{N}(s), \\
&\nabla_{\mathbf{T}(s)}\mathbf{N}(s) = \kappa_1(s)\mathbf{T}(s) + \kappa_2(s)\mathbf{B}(s), \\
&\nabla_{\mathbf{T}(s)}\mathbf{B}(s) = -\kappa_2(s)\mathbf{N},
\end{aligned} \quad (9)$$

where κ_1, κ_2 are curvature function and torsion function, respectively.

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned}\mathbf{T}(s) &= T_1(s)\,\mathbf{e}_1 + T_2(s)\,\mathbf{e}_2 + T_3(s)\,\mathbf{e}_3,\\ \mathbf{N}(s) &= N_1(s)\,\mathbf{e}_1 + N_2(s)\,\mathbf{e}_2 + N_3(s)\,\mathbf{e}_3,\\ \mathbf{B}(s) &= \mathbf{T}(s)\times\mathbf{N}(s) = B_1(s)\,\mathbf{e}_1 + B_2(s)\,\mathbf{e}_2 + B_3(s)\,\mathbf{e}_3.\end{aligned}$$

Theorem 3.1. *(see [15]) $\gamma : I \to Heis^3$ be a unit speed timelike biharmonic curve if and only if*

$$\begin{aligned}\kappa_1(s) &= \text{constant} \neq 0,\\ \kappa_1^2(s) - \kappa_2^2(s) &= 1 - 4B_1^2(s),\\ \kappa_2'(s) &= 2N_1(s)\,B_1(s).\end{aligned} \tag{10}$$

Theorem 3.2. *If $\gamma : I \to Heis^3$ is a unit speed timelike biharmonic curve, then γ is timelike helix.*

Proof. We can use (7) to compute the covariant derivatives of the vector fields $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ as:

$$\begin{aligned}\nabla_{\mathbf{T}(s)}\mathbf{T}(s) &= T_1'(s)\,\mathbf{e}_1 + (T_2'(s) + 2T_1(s)\,T_3(s))\mathbf{e}_2\\ &\quad + (T_3'(s) + 2T_1(s)\,T_2(s))\mathbf{e}_3,\\ \nabla_{\mathbf{T}(s)}\mathbf{N}(s) &= (N_1'(s) + T_2(s)\,N_3(s) - T_3(s)\,N_2(s))\mathbf{e}_1\\ &\quad + (N_2'(s) + T_1(s)\,N_3(s) + T_3(s)\,N_1(s))\mathbf{e}_2\\ &\quad + (N_3'(s) + T_2(s)\,N_1(s) + T_1(s)\,N_2(s))\mathbf{e}_3,\\ \nabla_{\mathbf{T}(s)}\mathbf{B}(s) &= (B_1'(s) + T_2(s)\,B_3(s) - T_3(s)\,B_2(s))\mathbf{e}_1\\ &\quad + (B_2'(s) + T_1(s)\,B_3(s) + T_3(s)\,B_1(s))\mathbf{e}_2\\ &\quad + (B_3'(s) + T_2(s)\,B_1(s) + T_1(s)\,B_2(s))\mathbf{e}_3.\end{aligned} \tag{11}$$

It follows that the first components of these vectors are given by

$$\begin{aligned}\langle\nabla_{\mathbf{T}(s)}\mathbf{T}(s),\mathbf{e}_1\rangle &= T_1'(s),\\ \langle\nabla_{\mathbf{T}(s)}\mathbf{N}(s),\mathbf{e}_1\rangle &= N_1'(s) + T_2(s)N_3(s) - T_3(s)N_2(s),\\ \langle\nabla_{\mathbf{T}(s)}\mathbf{B}(s),\mathbf{e}_1\rangle &= B_1'(s) + T_2(s)B_3(s) - T_3(s)B_2(s).\end{aligned} \tag{12}$$

On the other hand, using Frenet formulas (9), we have

$$\begin{aligned}\langle\nabla_{\mathbf{T}(s)}\mathbf{T}(s),\mathbf{e}_1\rangle &= \kappa_1 N_1(s),\\ \langle\nabla_{\mathbf{T}(s)}\mathbf{N}(s),\mathbf{e}_1\rangle &= \kappa_1 T_1(s) + \kappa_2(s)\,B_1,\\ \langle\nabla_{\mathbf{T}(s)}\mathbf{B}(s),\mathbf{e}_1\rangle &= -\kappa_2(s)\,N_1(s).\end{aligned} \tag{13}$$

These, together with (12) and (13), give

$$\begin{aligned}T_1'(s) &= \kappa_1 N_1(s),\\ N_1'(s) + T_2(s)N_3(s) - T_3(s)N_2(s) &=\\ &\kappa_1 T_1(s) + \kappa_2(s)B_1(s),\\ B_1'(s) + T_2(s)B_3(s) - T_3(s)B_2(s) &= -\kappa_2(s)N_1(s).\end{aligned} \tag{14}$$

Assume that γ is biharmonic.

If we take the derivative in the second equation of (10), we get

$$\kappa_2'(s)\,\kappa_2(s) = 4B_1(s)\,B_1'(s).$$

Then using $\kappa_2'(s) = 2N_1(s)\,B_1(s) \neq 0$ and (14), we obtain

$$\kappa_2(s)\,N_1(s)\,B_1(s) = 2B_1(s)\,B_1'(s).$$

Then,

$$\kappa_2(s) = \frac{2B_1'(s)}{N_1(s)}. \tag{15}$$

If we use $T_2(s)\,B_3(s) - T_3(s)\,B_2(s) = N_1(s)$ and (14), we get

$$B_1'(s) = (1 - \kappa_2(s))N_1(s).$$

We substitute $B_1'(s)$ in (15):

$$\kappa_2(s) = \frac{2}{3} = \text{constant}.$$

Therefore, also $\kappa_2(s)$ is constant and we have a contradiction that is $\kappa_2'(s) = 2N_1(s)\,B_1(s) \neq 0$. This completes the proof.

Corollary 3.3. *$\gamma : I \to Heis^3$ is a unit speed timelike biharmonic if and only if*

$$\begin{aligned}\kappa_1 &= \text{constant} \neq 0,\\ \kappa_2 &= \text{constant},\\ N_1(s)B_1(s) &= 0,\\ \kappa_1^2 - \kappa_2^2 &= 1 - 4B_1^2(s).\end{aligned} \tag{16}$$

Corollary 3.4. *(see [15]) Let $\gamma : I \to Heis^3$ be a timelike curve on Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 \neq 0$ then γ is not biharmonic.*

4. Canal Surfaces around Horizontal Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3

Now, we shall give here the mathematical description of canal surfaces associated to timelike horizontal biharmonic curves in the Lorentzian Heisenberg group Heis^3. Our purpose in this section, we will obtain the tubular surface from the canal surface in the Lorentzian Heisenberg group Heis^3. If we find the canal surface with taking variable radius $r(s)$ as constant, then the tubular surface can be found, since the canal surface is a general case of the tubular surface.

Firstly, consider a nonintegrable two-dimensional distribution $(x,y) \to \mathcal{H}_{(x,y)}$ in Heis^3 defined as $\mathcal{H} = \ker\omega$, where $\omega = x\mathrm{d}y + \mathrm{d}z$ is a 1-form on Heis^3. The distribution $\mathcal{H}$ is called the horizontal distribution.

A curve $\gamma : I \to \mathrm{Heis}^3$ is called horizontal curve if $\gamma'(s) \in \mathcal{H}_{\gamma(s)}$, for every s.

Lemma 4.1. *Let $\gamma : I \to \mathrm{Heis}^3$ is a timelike horizontal curve. Then,*

$$z'(s) + x(s)y'(s) = 0. \tag{17}$$

Proof. Using the orthonormal left-invariant frame (7), we have

$$\begin{aligned}\gamma'(s) &= x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z \\ &= x'(s)\mathbf{e}_3 + y'(s)\mathbf{e}_2 + \omega(\gamma'(s))\mathbf{e}_1.\end{aligned}$$

Then, $\gamma(s)$ is a timelike horizontal curve, we get

$$\omega(\gamma'(s)) = 0.$$

We substitute $\omega = x\mathrm{d}y + \mathrm{d}z$ in the above equation

$$\omega(\gamma'(s)) = z'(s) + x(s)y'(s) = 0. \tag{18}$$

We obtain (17) and the lemma is proved.

Lemma 4.2. *If $\gamma(s)$ is a timelike horizontal curve, then*

$$\begin{aligned}x'(s)\mathbf{e}_3 + y'(s)\mathbf{e}_2 &= x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} \\ &\quad - x(s)y'(s)\frac{\partial}{\partial z}.\end{aligned} \tag{19}$$

Proof. Using our orthonormal basis, we obtain

$$\frac{\partial}{\partial x} = e_3, \quad \frac{\partial}{\partial y} = e_2 + xe_3, \quad \frac{\partial}{\partial z} = e_1.$$

Substituting above system in Lemma 4.1, we have (19).

On the other hand, an envelope of a 1-parameter family of surfaces is constructed in the same way that we constructed a 1-parameter family of curves. The family is described by a differentiable function $F(x,y,z,\lambda) = 0$, where λ is a parameter. When λ can be eliminated from the equations

$$F(x,y,z,\lambda) = 0$$

and

$$\frac{\partial F(x,y,z,\lambda)}{\partial\lambda} = 0.$$

We get the envelope, which is a surface described implicitly as $G(x,y,z) = 0$. For example, for a 1-parameter family of planes we get a develople surface [23].

Definition 4.3. *The envelope of a 1-parameter family of the Lorentzian spheres in the Lorentzian Heisenberg group Heis^3 is called a canal surface in the Lorentzian Heisenberg group Heis^3. The curve formed by the centers of the Lorentzian spheres is called center curve of the canal surface. The radius of the canal surface is the function r such that $r(s)$ is the radius of the Lorentzian sphere. Here the Lorentzian circle is in the plane determined by $\gamma(s)$, $\mathbf{N}(s)$, $\mathbf{B}(s)$ and with its center in $\gamma(s)$.*

On the other hand, let $\gamma : I \to \mathrm{Heis}^3$ be a unit speed curve whose curvature does not vanish. Consider a tube of radius r around γ. Since the normal $\mathbf{N}(s)$ and binormal $\mathbf{B}(s)$ are perpendicular to γ, the Lorentzian circle is perpendicular γ and $\gamma(s)$. As this Lorentzian circle moves along γ, it traces out a surface about γ which will be the tube about γ, provided r is not too large.

Theorem 4.4. *Let the center curve of a canal surface $\mathfrak{Canal}(s,\theta)$ be a unit speed timelike horizontal biharmonic curve $\gamma : I \to \mathrm{Heis}^3$. Then, the parametric equations of $\mathfrak{Canal}(s,\theta)$ are*

$$\begin{aligned}&x_{\mathfrak{Canal}}(s,\theta) = \\ &\frac{1}{\kappa_1}\sinh(\kappa_1 s + \zeta) + r(s)r'(s)\cosh(\kappa_1 s + \zeta) \\ &\pm r(s)\sqrt{1 + (r'(s))^2}\sinh(\kappa_1 s + \zeta)\cos\theta + a_1,\end{aligned}$$

$$
\begin{aligned}
&y_{\mathfrak{Canal}}(s,\theta) = \\
&\frac{1}{\kappa_1}\cosh(\kappa_1 s+\zeta)+r(s)r'(s)\sinh(\kappa_1 s+\zeta)\\
&\pm r(s)\sqrt{1+(r'(s))^2}\cosh(\kappa_1 s+\zeta)\cos\theta+a_2,\\
&z_{\mathfrak{Canal}}(s,\theta) = \\
&\frac{2}{\kappa_1}s-\frac{1}{4\kappa_1^2}\sinh 2(\kappa_1 s+\zeta)-\frac{a_1}{2\kappa_1}\cosh(\kappa_1 s+\zeta)\\
&+r(s)r'(s)\left[-\frac{1}{\kappa_1}\sinh^2(\kappa_1 s+\zeta)-a_1\sinh(\kappa_1 s+\zeta)\right]\\
&\pm r(s)\sqrt{1+(r'(s))^2}\cosh(\kappa_1 s+\zeta)\\
&\cdot\left[-\frac{1}{\kappa_1^2}\sinh(\kappa_1 s+\zeta)-c_1 s-c_2\right]\cos\theta\\
&\pm r(s)\sqrt{1+(r'(s))^2}\sin\theta+a_3,
\end{aligned}
\tag{20}
$$

where a_1,a_2,a_3,c_1,c_2 are constants of integration and $r(s)$ is the radius of the Lorentzian sphere.

Proof. Since γ is timelike biharmonic, γ is a timelike helix. So, without loss of generality, we take the axis of γ parallel to the spacelike vector $\mathbf{e}_1$. Then,

$$g(\mathbf{T(s)},\mathbf{e}_1)=T_1(s)=\sinh\varphi, \tag{21}$$

where φ is a constant angle.

The tangent vector can be written in the following form:

$$\mathbf{T}(s)=T_1(s)\mathbf{e}_1+T_2(s)\mathbf{e}_2+T_3(s)\mathbf{e}_3. \tag{22}$$

On the other hand, the tangent vector $\mathbf{T}$ is a unit timelike vector, so the following condition is satisfied:

$$T_2^2(s)-T_3^2(s)=-1-\sinh^2\varphi. \tag{23}$$

Noting that $\cosh^2\varphi-\sinh^2\varphi=1$, we have

$$T_3^2(s)-T_2^2(s)=\cosh^2\varphi. \tag{24}$$

The general solution of (24) can be written in the following form:

$$
\begin{aligned}
T_2(s)&=\cosh\varphi\sinh\mu(s),\\
T_3(s)&=\cosh\varphi\cosh\mu(s),
\end{aligned}
\tag{25}
$$

where μ is an arbitrary function of s.

So, substituting the components $T_1(s)$, $T_2(s)$, and $T_3(s)$ in (22), we have the following equation:

$$
\begin{aligned}
\mathbf{T}(s)=&\sinh\varphi\mathbf{e}_1+\cosh\varphi\sinh\mu(s)\mathbf{e}_2\\
&+\cosh\varphi\cosh\mu(s)\mathbf{e}_3.
\end{aligned}
\tag{26}
$$

Since $|\nabla_{\mathbf{T}(s)}\mathbf{T}(s)|=\kappa_1$, we obtain

$$\mu(s)=\left(\frac{\kappa_1-\sinh 2\varphi}{\cosh\varphi}\right)s+\zeta, \tag{27}$$

where $\zeta\in\mathbb{R}$.

Thus, (26) and (27) imply

$$
\begin{aligned}
\mathbf{T}(s)=&\sinh\varphi\mathbf{e}_1+\cosh\varphi\sinh(\wp s+\zeta)\mathbf{e}_2\\
&+\cosh\varphi\cosh(\wp s+\zeta)\mathbf{e}_3,
\end{aligned}
\tag{28}
$$

where $\wp=\frac{\kappa_1-\sinh 2\varphi}{\cosh\varphi}$.

Using (5) in (27), we obtain

$$
\begin{aligned}
\mathbf{T}(s)=\Big(&\cosh\varphi\cosh(\wp s+\zeta),\ \cosh\varphi\sinh(\wp s+\zeta),\\
&\sinh\varphi-\frac{1}{\wp}\cosh^2\varphi\sinh^2(\wp s+\zeta)\\
&-a_1\cosh\varphi\sinh(\wp s+\zeta)\Big),
\end{aligned}
\tag{29}
$$

where a_1 is constant of integration.

From (29), the parametric equations of unit speed timelike biharmonic curve γ are

$$
\begin{aligned}
x_\gamma(s)=&\frac{1}{\wp}\cosh\varphi\sinh(\wp s+\zeta)+a_1,\\
y_\gamma(s)=&\frac{1}{\wp}\cosh\varphi\cosh(\wp s+\zeta)+a_2,\\
z_\gamma(s)=&\sinh\varphi s-\frac{1}{\wp}\cosh^2\varphi\left[-\frac{s}{2}+\frac{\sinh 2(\wp s+\zeta)}{4\wp}\right]\\
&-\frac{a_1\cosh\varphi}{\wp}\cosh(\wp s+\zeta)+a_3,
\end{aligned}
\tag{30}
$$

where a_1,a_2,a_3 are constants of integration.

On the other hand, using (18) and (28) we have

$$T_1(s)=\sinh\varphi=0. \tag{31}$$

Thus, we choose

$$\cosh\varphi=1. \tag{32}$$

Using (31) and (32) in the system (30), then the parametric equations of unit speed timelike horizontal biharmonic curve γ are

$$
\begin{aligned}
x_\gamma(s)=&\frac{1}{\kappa_1}\sinh(\kappa_1 s+\zeta)+a_1,\\
y_\gamma(s)=&\frac{1}{\kappa_1}\cosh(\kappa_1 s+\zeta)+a_2,\\
z_\gamma(s)=&-\frac{1}{\kappa_1}\left[-\frac{s}{2}+\frac{\sinh 2(\kappa_1 s+\zeta)}{4\kappa_1}\right]\\
&-\frac{a_1}{\kappa_1}\cosh(\kappa_1 s+\zeta)+a_3,
\end{aligned}
\tag{33}
$$

where a_1,a_2,a_3 are constants of integration.

Substituting (31) into (28), we get

$$\mathbf{T}(s) = \sinh(\kappa_1 s + \zeta)\mathbf{e}_2 + \cosh(\kappa_1 s + \zeta)\mathbf{e}_3. \quad (34)$$

Using (5) in (34), we obtain

$$\mathbf{T}(s) = \Big(\cosh(\kappa_1 s + \zeta),\ \sinh(\kappa_1 s + \zeta), -\frac{1}{\kappa_1}\sinh^2(\kappa_1 s + \zeta) - a_1 \sinh(\kappa_1 s + \zeta)\Big). \quad (35)$$

Assume that the center curve of a tubular surface is a unit speed timelike biharmonic curve γ and $\mathfrak{Canal}(s,\theta)$ denote a patch that parametrizes the envelope of the Lorentzian spheres defining the tubular surface. Then we obtain

$$\mathfrak{Canal}(s,\theta) = \gamma(s) + \xi(s,\theta)\mathbf{T}(s) + \eta(s,\theta)\mathbf{N}(s) + \rho(s,\theta)\mathbf{B}(s), \quad (36)$$

where ξ, η, and ρ are differentiable on the interval on which γ is defined.

On the other hand, using Frenet formulas (9) and (34), we have

$$\mathbf{N}(s) = \cosh(\kappa_1 s + \zeta)\mathbf{e}_2 + \sinh(\kappa_1 s + \zeta)\mathbf{e}_3].$$

Similarly, using (5) in above equation, we obtain

$$\mathbf{N}(s) = \Big(\sinh(\kappa_1 s + \zeta),\ \cosh(\kappa_1 s + \zeta), -\frac{1}{\kappa_1^2}\sinh(\kappa_1 s + \zeta)\cosh(\kappa_1 s + \zeta) - c_1 s\cosh(\kappa_1 s + \zeta) - c_2\cosh(\kappa_1 s + \zeta)\Big). \quad (37)$$

On the other hand, the binormal vector $\mathbf{B}(s)$ is

$$\mathbf{B}(s) = -\mathbf{e}_1 = (0,0,-1). \quad (38)$$

Using Definition 4.3, we have

$$g\big(\mathfrak{Canal}(s,\theta) - \gamma(s),\ \mathfrak{Canal}(s,\theta) - \gamma(s)\big) = r^2(s). \quad (39)$$

Since $C(s,\theta) - \gamma(s)$ is a normal vector to the tubular surface, we get

$$g(\mathfrak{Canal}(s,\theta) - \gamma(s),\ \mathfrak{Canal}_s(s,\theta)) = 0. \quad (40)$$

From (18) and (35), we get

$$\begin{aligned} -\xi^2(s) + \eta^2(s) + \rho^2(s) &= r^2(s), \\ -\xi(s)\xi_s(s) + \eta(s)\eta_s(s) + \rho(s)\rho_s(s) &= r(s)r'(s). \end{aligned} \quad (41)$$

When we differentiate (36) with respect to s and use the Frenet–Serret formulas, we obtain

$$\begin{aligned} \mathfrak{Canal}_s(s,\theta) &= \big(1 + \xi_s(s) + \eta(s)\kappa_1\big)\mathbf{T}(s) \\ &\quad + \big(\xi(s)\kappa_1 - \rho(s)\kappa_2 + \eta_s(s)\big)\mathbf{N}(s) \\ &\quad + \big(\rho_s(s) + \eta(s)\kappa_2\big)\mathbf{B}(s). \end{aligned} \quad (42)$$

Then (40), (41), and (42) imply that

$$\xi(s) = r(s)r'(s). \quad (43)$$

Also, from (41) and (42) we get

$$\eta^2(s) + \rho^2(s) = r^2(s)\big(1 + (r'(s))^2\big). \quad (44)$$

The solution of (44) can be written in the following form:

$$\begin{aligned} \eta(s) &= \pm r(s)\sqrt{1 + (r'(s))^2}\cos\theta, \\ \rho(s) &= \pm r(s)\sqrt{1 + (r'(s))^2}\sin\theta. \end{aligned} \quad (45)$$

Thus (36) becomes

$$\begin{aligned} \mathfrak{Canal}(s,\theta) &= \gamma(s) + r(s)r'(s)\mathbf{T}(s) \\ &\quad \pm r(s)\sqrt{1 + (r'(s))^2}\mathbf{N}(s)\cos\theta \\ &\quad \pm r(s)\sqrt{1 + (r'(s))^2}\mathbf{B}(s)\sin\theta. \end{aligned} \quad (46)$$

Substuting (33), (35), (37), and (38) into (46), we obtain the system (20). This completes the proof.

Corollary 4.5. *If the radius of the Lorentzian sphere is $r(s) = s$, then the parametric equations $\mathfrak{Canal}(s,\theta)$ are*

$$\begin{aligned} x_{\mathfrak{Canal}}(s,\theta) &= \frac{1}{\kappa_1}\sinh(\kappa_1 s + \zeta) + s\cosh(\kappa_1 s + \zeta) \\ &\quad \pm \sqrt{2}s\sinh(\kappa_1 s + \zeta)\cos\theta + a_1, \\ y_{\mathfrak{Canal}}(s,\theta) &= \frac{1}{\kappa_1}\cosh(\kappa_1 s + \zeta) + s\sinh(\kappa_1 s + \zeta) \\ &\quad \pm \sqrt{2}s\cosh(\kappa_1 s + \zeta)\cos\theta + a_2, \end{aligned}$$

$$
\begin{aligned}
z_{\mathfrak{Canal}}(s,\theta) = & \frac{2}{\kappa_1}s - \frac{1}{4\kappa_1^2}\sinh 2(\kappa_1 s+\zeta) \\
& - \frac{a_1}{2\kappa_1}\cosh^2(\kappa_1 s+\zeta) \qquad (47) \\
& + s\left[-\frac{1}{\kappa_1}\sinh^2(\kappa_1 s+\zeta) - a_1\sinh(\kappa_1 s+\zeta)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \pm\sqrt{2}s\sin\theta \pm \sqrt{2}s\cosh(\kappa_1 s+\zeta) \\
& \cdot\left[-\frac{1}{\kappa_1^2}\sinh(\kappa_1 s+\zeta) - c_1 s - c_2\right]\cos\theta + a_3,
\end{aligned}
$$

where a_1, a_2, a_3, c_1, c_2 are constants of integration.

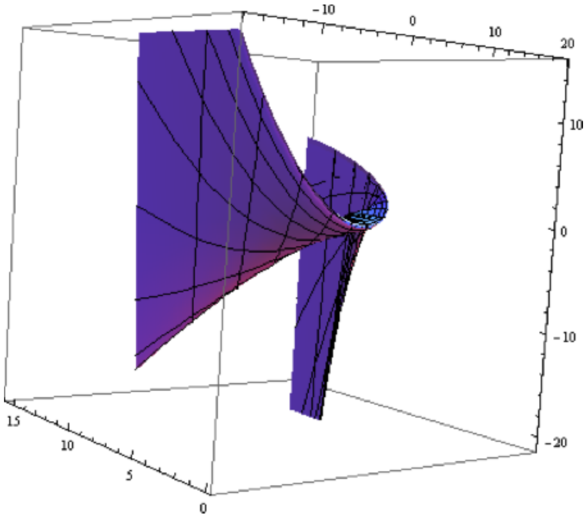

Fig. 1 (colour online). Plot with the parameters $\zeta = 0$ and $\kappa_1 = r = a_1 = a_2 = a_3 = c_1 = c_2 = 1$.

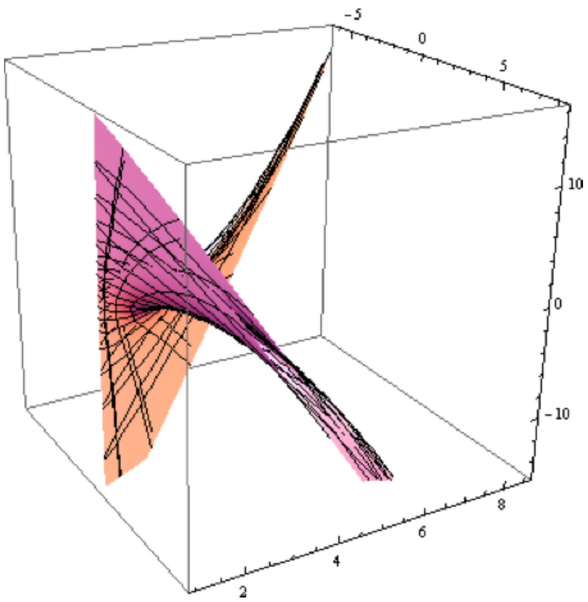

Fig. 2 (colour online). Plot with the parameters $\zeta = 0$ and $\kappa_1 = r = a_1 = a_2 = a_3 = c_1 = c_2 = 1$.

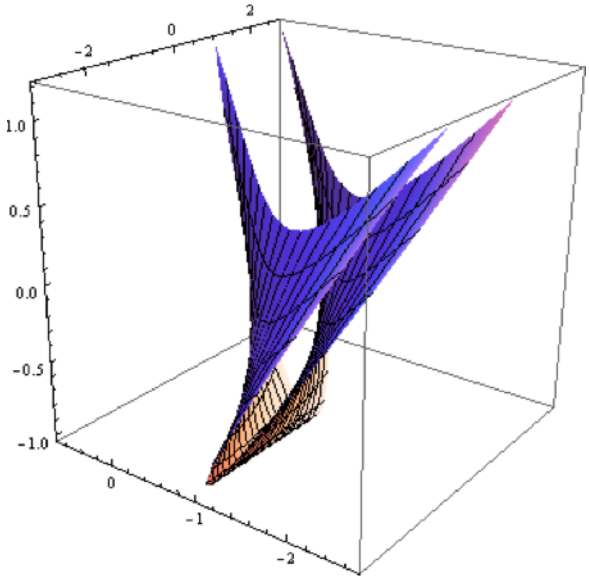

Fig. 3 (colour online). Plot with the parameters $\kappa_1 = 1$, $\zeta = 0$ and $r = a_1 = a_2 = a_3 = c_1 = c_2 = -1$.

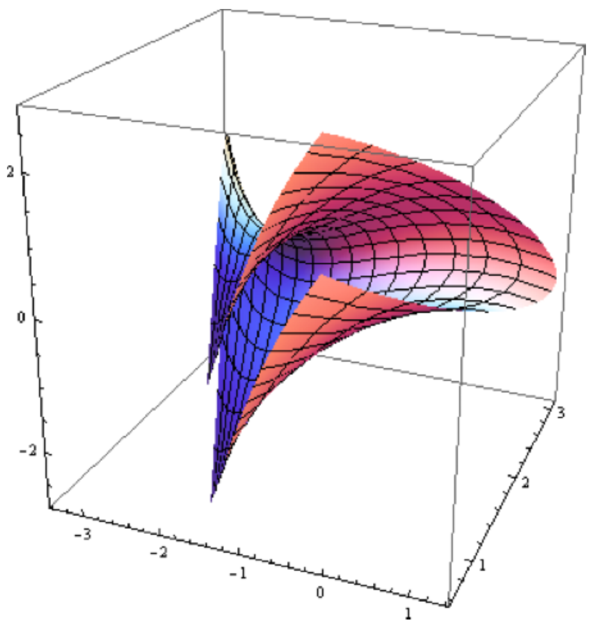

Fig. 4 (colour online). Plot with the parameters $\kappa_1 = r = 1$, $\zeta = 0$ and $a_1 = a_2 = a_3 = c_1 = c_2 = 0$.

It is easy to see that when the radius function $r(s)$ is constant, the definition of a canal surface reduces to the definition of a tube. In fact, we can characterize tubes among all canal surfaces.

Theorem 4.6. *If $\mathfrak{Canal}(s,\theta)$ is a tubular surface, that is $r(s)$ is constant, then the parametric equations of tubular surface are*

$$\begin{aligned} x_{\mathfrak{Tube}}(s,\theta) &= \frac{1}{\kappa_1}\sinh(\kappa_1 s+\zeta) \\ &\quad + r\sinh(\kappa_1 s+\zeta)\cos\theta + a_1, \\ y_{\mathfrak{Tube}}(s,\theta) &= \frac{1}{\kappa_1}\cosh(\kappa_1 s+\zeta) \\ &\quad + r\cosh(\kappa_1 s+\zeta)\cos\theta + a_2, \\ z_{\mathfrak{Tube}}(s,\theta) &= \frac{2}{\kappa_1}s - \frac{1}{4\kappa_1^2}\sinh 2(\kappa_1 s+\zeta) \\ &\quad - \frac{a_1}{2\kappa_1}\cosh^2(\kappa_1 s+\zeta) + r\sin\theta + r\cosh(\kappa_1 s+\zeta) \\ &\quad \cdot \left[-\frac{1}{\kappa_1^2}\sinh(\kappa_1 s+\zeta) - c_1 s - c_2\right]\cos\theta + a_3, \end{aligned} \tag{48}$$

where a_1, a_2, a_3, c_1, c_2 are constants of integration.

Next, we apply Theorem 4.6.

We can draw the tubular surface $\mathfrak{Tube}(s,\theta)$ with the help of the programme Mathematica as it can be seen in Figures 1–4.

Acknowledgements

The authors thank to the referee for useful suggestions and remarks for the revised version.

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